Journal of Global Optimization (2005) 32: 507–516
DOI 10.1007/s10898-003-2687-y

Full Nuclear Cones and a Relation Between Strong Optimization and Pareto Efficiency

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(Received 15 August 2003; accepted 19 August 2003)

Abstract. In this paper we present some new and pertinent connections between the Strong Optimization and the Approximate Pareto type Efficiency, in particular, with the usual Vector Optimization, at first in the Ordered Vector Spaces by the natural Convex Cones and, afterwards, in the Ordered Hausdorff Locally Convex Spaces. The main result is obtained considering the notion of full nuclear cone. Our results, is related to an appropriate scalarization method.

Mathematics Subject Classifications (2000). 90C29.

Key words: full nuclear cone, Pareto efficiency, supernormal (nuclear) cone.

1. Introduction

One of the main research direction concerning in Vector Optimization is the study of the existence of Pareto type efficient points (see, for instance, the recent results, comments and examples given in [7–15, 22–26, 30,31], among others). In this context, it can be immediately remarked very clear that an important mathematical tool imposed especially by its implications and applications to the study of this kind of optimality was, and it remains actual the concept of nuclear (supernormal) cone introduced by G. Isac in [7], published in [8] (with a solid motivation regarding, at least the Vector Optimiztion) and developed With significant applications until now in other connected papers.

Recently, G. Isac introduced in [15] the notion of *full nuclear cone*. The reader can find in [15] also some properties of this class of convex cones, locally convex spaces.

The main result in [15] is a general necessary and sufficient test for Pareto efficiency. This test is based, on the notion of full nuclear cone.

Now, in this paper we present an improvement of this general test. By this improvement we present a connection between the strong optimization and Pareto efficiency.

We obtained this result being inspired by the notion of full nuclear cone. Perhaps, the results presented in this paper may be the origin of new investigation in the Vector Optimization Theory.

2. Preliminaries

We denote in this paper by X a real vector space or a locally convex vector space. We use for a locally convex vector space, the definition given by Treves [29] i.e., a locally convex space is a couple (X, Spec(X)), where X is a real vector space and Spec(X) is a family of seminorms on X satisfying the following properties:

(1) $\lambda p \in \mathrm{Spec}(X)$, whenever $\lambda \in R_+$ and $p \in \mathrm{Spec}(X)$,
(2) if $p \in \mathrm{Spec}(X)$ and q is a seminorm on X such that $q \leqslant p$, then $q \in \mathrm{Spec}(X)$,
(3) for every $p_1, p_2 \in \mathrm{Spec}(E), \sup(p_1, p_2) \in \mathrm{Spec}(X)$ where $\sup(p_l, p_2)(x) = \sup(p_1(x), p_2(x))$, for any $x \in X$.

It is known [29] that if Spec(X) is given, then there exists a locally convex topology τ on X such that $X(\tau)$ is a topological vector space, such that a seminorm p on X is τ-continuous if and only if $p \in$ Spec(X).

We say that a subset $\mathcal{B} \subset \mathrm{Spec}(X)$ is abase of Spec(X) if and only if, for every $p \in \mathrm{Spec}(X)$ there exists $q \in \mathcal{B}$ and a real number $\lambda > 0$ such that $p \leqslant \lambda q$.

The topology τ defined on X by Spec(X) is a Hausdorff topology if Spec(X) has a base $\mathcal{B}$ satisfying the following property:

$$\{x \in X \mid p(x) = 0, \text{ for all } p \in \mathcal{B}\} = \{0\}.$$

In this case we say that $\mathcal{B}$ is a *Hausdorff* base for Spec(X). We denote X^* the topological dual of X. Let X be a real vector space. We say that a non-empty subset K of X is a convex cone if the following properties are satisfied:

(k_1) $K + K \subseteq K$,
(k_2) $\lambda K \subseteq K$, for every $\lambda \in R_+$.
A convex cone $K \subset X$ is said to be *pointed* if the following property is satisfied:
(k_3) $K \cap (-K) = \{0\}$.

When X is a locally convex vector space and $K \subset X$ is a convex one, we suppose also that K is a closed set in X. If a convex cone $K \subset X$ is given, we denote by K^* the dual of K i.e., $K^* = \{x^* \in X^* : x^*(x) \geqslant 0, \text{ for all } x \in K\}$, and by K^0 the polar of K, i.e., $K^0 = -K^*$.

If τ is the topology defined by Spec(X), we recall that a pointed convex cone $K \subset X$ is *normal* (with respect to the topology τ), if and only if, one of the following equivalent assertions is satisfied:

(n_1) there exists a base $\mathcal{B}$ of Spec(X) such that for every $p \in \mathcal{B}$ and every $x, y \in K$, such that $x \leqslant y$ we have $p(x) \leqslant p(y)$,
(n_2) if $\{x_i\}_{i\in I}, \{y_i\}_{i\in I}$ are two arbitrary nets in K, such that for every $i \in I, 0 \leqslant x_i \leqslant y_i$ and $\lim_{i\in I} y_i = 0$, then we have $\lim_{i\in I} x_i = 0$.

We note that the notion of *normal cone* is the most important notion in the theory of convex cones in topological vector spaces.

Let X be a vector space ordered by a convex cone, K and let K_1 be a non-empty subset of K and A a non-empty subset of X. The following definition introduces a new concept of (approximate) Pareto type efficient points which, particularly, leads to the well know notion of Pareto efficiency (in fact, the generalization in abstract spaces of the finite dimensional notion).

DEFINITION 2.1. We say that $a_0 \in A$ is a K_1-Pareto (minimal) efficient point of A, in notation, $a_0 \in \text{eff}(A, K, K_1)$ (or $a_0 \in \text{MIN}_{k+k1}(A)$) if it satisfies one of the following equivalent conditions:

(i) $A \cap (a_0 - K - K_1) \subseteq a_0 + K + K_1$
(ii) $(K + K_1) \cap (a_0 - A) \subseteq -K - K_1$
(iii) $(K + K_1) \cap (a_0 - A - K - K_1) \subseteq -K - K_1$

In a similar manner one defines the Pareto (maximal) efficient points by replacing $K + K_1$ with $-(K + K_1)$.

REMARK 2.1. $a_0 \in \text{eff}(A, K, K_1)$ if it a fixed point for one of the following multifunctions $F_i\colon A \to A, i \in \{1, 2, 3, 4\}$ defined by

$$F_1(t) = \{a \in A : A \cap (a - K - K_1) \subseteq t + K + K_1\}$$
$$F_2(t) = \{a \in A : A \cap (t - K - K_1) \subseteq a + K + K_1\}$$
$$F_3(t) = \{a \in A : (A + K + K_1) \cap (a - K - K_1) \subseteq t + K + K_1\}$$
$$F_4(t) = \{a \in A : (A + K + K_1) \cap (t - K - K_1) \subseteq a + K + K_1\}$$

Consequently, for the existence of the Pareto type efficient points it can applied appropriate fixed point theorems concerning the multi-functions (see, for instance, [4] and other recent papers).

REMARK 2.2. In [20] it is shown that whenever $K_1 \subset K \backslash \{0\}$, the existence of this new type of efficient points for lower bounded sets characterizes the semi-Archimedian ordered vector spaces and the regular ordered locally convex spaces.

REMARK 2.3. When K is pointed, that is, $K \cap (-K) = \{0\}$ and $K_1 = \{0\}$, then, from Definition 2.1, we obtain the well-known usual notion of Pareto (minimal, efficient, optimal or admissible) point, abbreviated by

$$a_0 \in \text{eff}(A, K) = \text{eff}(A, K, \{0\}), \quad (\text{or } a_0 \in \text{MIN}_K(A)),$$

that is satisfying the next equivalent properties:

(i) $A \cap (a_0 - K) = \{a_0\}$;
(ii) $A \cap (a_0 - K/\{0\}) = \varnothing$
(iii) $K \cap (a_0 - A) = \{0\}$;
(iv) $(K\backslash\{0\}) \cap (a_0 - A) = \varnothing$

and it is clear that for any $\varepsilon \in K\backslash\{0\}$, taking $K_1 = \{\varepsilon\}$, it follows that $a_0 \in \text{eff}(A, K, K_1)$ if and only if $A \cap (a_0, -\varepsilon - K) = \varnothing$. In all these cases, the set $\text{eff}(A, K, K_1)$ was denoted by $\varepsilon - \text{eff}(A, K)$(or $\varepsilon - \text{MIN}_K(A)$ as in [20, 25] and [13, 26]) and it is obvious that $\text{eff}(A, K) = \bigcap_{i \in K\backslash\{0\}} [\varepsilon - \text{eff}(A, K)]$

We put also in evidence the following facts.

(a) If $0 \notin K_1$ then *Definition* 2.1 is equivalent to the following condition: $A \cap (a_0 - K - K_1) = \phi$ if and only if $(K + K_1) \cap (a_0 - A) = \phi$.
(b) If $0 \in K_1$ then *Definition* 2.1 is equivalent to the condition:

$$A \cap (a_0 - K - K_1) = \{a_0\}.$$

(c) If $K \cap (-K_1) = \{0\}$ then we have the following relation $\text{eff}(A, K, K_1) = \text{eff}(A, K) = \bigcap_{[0] \neq K, \subset K} \text{eff}(A, K, K_2)$.

3. Nuclear and Full Nuclear Cones

Let $(X(\tau), \text{Spec}(X))$ be a locally convex space and $K \subset X$ a pointed convex cone. We recall the following notion.

DEFINITION 3.1. (Sec [8]). We say that K is a nuclear cone (with respect to the topology τ), if and only if there exists a base $\mathcal{B} = \{p_i\}_{i \in I}$ of $\text{Spec}(X)$ such that for any $p \in \mathcal{B}$, there exists $f_p, \in K^*$ such that $p(x) \leqslant f_p(x)$, for every $x \in K$.

About this notion the reader is referred to [7–18], [1–14], [23–26] and [31, 32]. In the papers cited above are presented examples and properties of nuclear cones and also applications to Pareto optimization and to Functional Analysis.

We note that in every locally convex space, every normal cone is nuclear with respect to the weak topology. The notion of nuclear cone is more interesting in locally convex spaces than in normed vector space, since in any normed vector space a pointed convex cone nuclear if and only if it is well-based [8].

Let $\mathcal{B}$ be a Housdorff base of Spec(X). Obviously, we can have $\mathcal{B} =$ Spec(X). Suppose given a pointed closed convex cone $K \subset X$. Let $\varphi : \mathcal{B} \to K^* \backslash \{0\}$ be an arbitrary mapping. We define the set $K_\varphi = \{x \in X : p(x) \leqslant \varphi(p)(x)$, for all $p \in \mathcal{B}$.

We proved in [15] that if $K_\varphi \neq \{0\}$, then K_φ, is a closed pointed convex cone. It is also known, that if K is a closed pointed and normal cone, then there exists a mapping $\varphi : \mathcal{B} \to K^* \backslash \{0\}$ such that $K_\varphi \neq \{0\}$. We note also the following result.

PROPOSITION 3.1. *A closed pointed convex cone $K \subset X$ is nuclear if and only if there exists a mapping $\varphi : \mathcal{B} \to K^* \backslash \{0\}$ such that $K \subseteq K_\varphi$.*

Proof. First, we note that we remarked in [15] that if K is nuclear, then there exists a mapping $\varphi : \mathcal{B} \to K^* \backslash \{0\}$ such that $K \subseteq K_\varphi$. We show now that the converse is also true. Indeed let $p \in \mathcal{B}$ be an arbitrary seminorm. We take $f_p = \varphi(p) \in K^* \backslash \{0\}$. Since $K \subseteq K_\varphi$, we have that for every $x \in K$, $p(x) \leqslant \varphi(p)(x) = f(x)$, that if K is a nuclear cone.

When $K_\varphi \neq \{0\}$ it is called *full nuclear* cone associated to the cone K. □

Using the notion of *full nuclear cone*, we proved in [15] the following result, related to the existence of Pareto efficient points.

THEOREM 3.1 (See [15]). *Let $(X,$ Spec$(X))$ be a Hausdorff locally convex space, $K \subset X$ a closed pointed convex cone and $A \subset X$ a non-empty subset. The set A has a Pareto (minimal) efficient point with respect to K if and only if, there exists an element $u_0 \in A$, a Hausdorff base $\mathcal{B}$ of* Spec(X)*, a mapping $\varphi : \mathcal{B} \to K^* \backslash \{0\}$, and an element $x* \in D = A \cap (u_0 - K)$ such that $D - x* \subset K_\varphi$ (i.e., x^* is a least element of the set D with respect to the full nuclear cone K_φ).*

In the next section we will give an improvement of this result.

4. The Main Results and Related Topics

The following theorem offers the first important connection between the strong optimization and the (approximate) Pareto efficiency in the context of ordered vector spaces, as it was described initially in the previous Definition 2.1.

THEOREM 4.1. *If we denote by* $S(A, K, K_1) = \{a_1 \in A: \subseteq a_1 + K + K_1\}$ *and* $S(A, K, K_1) \neq \varnothing$*, then* $S(A, K, K_1) = \mathrm{eff}(A, K, K_1)$.

Proof. Clearly, $S(A, K, K_1) \subseteq \mathrm{eff}(A, K, K_1)$. Indeed, if $a_0 \in S(A, K, K_1)$ and $a \in A \cap (a_0 - K - K_1)$ are arbitrary elements, then $a \in a_0 + K + K_l$, that is, $a_0 \in \mathrm{eff}(A, K, K_1)$, by virtue of (i) in Definition 2.1. Suppose now that $\bar{a} \in S(A, K, K_1) \neq \varnothing$ and there exists $a_0 \in \mathrm{eff}(A, K, K_1) \setminus S(A, K, K_1)$. From $\bar{a} \in S(A, K, K_1)$ it follows that $a_0 \in \bar{a} + K + K_1$, that is $\bar{a} \in a_0 - K - K_1$, from which, since $\bar{a} \in A$ and $a_0 \in \mathrm{eff}(A, K, K_1)$ we conclude that $\bar{a} \in a_0 + K + K_1$. Therefore, $A \subseteq \bar{a} + K + K_1 \subseteq a_0 + K + K_1$, in contradiction with $a_0 \notin S(A, K, K_1)$ as claimed. □

We note that when $K = K_1$ we denote $S(A, K, K_1)$ by $S(A, K)$.

REMARK 4.1. The above theorem shows that, for any non-empty subset of an arbitrary vector space, the set of all strong minimal elements with respect to any convex cone through the agency of every non-empty subset of it coincides with the corresponding set of Pareto (minimal) efficient points whenever there exists at least a strong minimal element, the result remaining obviously valid for the strong maximal elements and the Pareto maximal efficient points, respectively.

Using this result and our abstract construction given in [22] for the H-locally convex spaces introduced by Th. Precupanu in [27] (as separated locally convex spaces with any seminorm satisfying the parallelogram law), we established in [25] that the only best simultaneous and vectorial approximation for each element in the direct sum of a (closed) linear subspace and its orthogonal with respect to a linear (continuous) operator between two H-locally convex spaces is its spline function. We also note that it is possible to have $S(A, K, K_1) = \varnothing$ and $\mathrm{eff}(A, K, K_1) = A$. Thus, for example, in the case when $X = R^2$ is endowed with the separated locally convex topology generated by the seminorms p_1, p_2 defined by

$$p_1, p_2: X \to R_+, \quad p_1(x, y) = |x|, \quad p_2(x, y) = |y|,$$
$$K = R^2_+ = \{(x, y) \in R^2: x, y \geqslant 0\}, K_1 = \{(0, 0)\}$$

and

$$A = \{(\lambda, 1 - \lambda: 0 \leqslant \lambda \leqslant 1)\}.$$

In this case it is clear that $S(A, K, K_1) = \varnothing$ and $\mathrm{eff}(A, K, K_1) = A$.

In all of our further considerations we suppose that X is a Hausdorff locally convex space having the topology induced by $\mathrm{Spec}(X) = P = \{p_a : \alpha \in I\}$ of semi-norms, and ordered by a convex cone K. We note again the topological dual of X by X^*. In this framework the next theorem contains a significant criterion for the existence of the approximate Pareto (minimal) efficient points, in particular, for the usual Pareto (minimal) efficient points, taking into account the dual cone K^* of K and its polar cone $K^0 = -K^*$. The version for the (approximate) Pareto (maximal) efficient points is straightforward.

THEOREM 4.2. *If A is any non-empty subset of X and K_i is an arbitrary non-empty subset of K, then $a_0 \in \mathrm{eff}(A, K, K_1)$ whenever for each $p_\alpha \in P$ and $\eta \in (0,1)$ there exists u^* in the polar cone K^0 of K such that $p_a(a_0 - a) \leqslant x^*(a_0 - a) + \eta, \forall a \in A$.*

Proof. We follow the general lines of the proof for Theorem 2.5 in [25]. Let us suppose that, under the above hypotheses, $(K + K_1) \cap (a_0 - A) \not\subset -(K + K_1)$ that is, there exists $a \in$ A such that $a_0 - a \in K + K_1 \backslash (-K - K_1)$. Then, $a_0 - a \neq 0$ and, because X is a Hausdorff space there exists $p_\alpha \in P$ such that $p_\alpha(a_0 - a) > 0$. On the other hand, there exists $n \in N^*$ sufficiently large with $p_a(a_0 - a)/n \in (0,1)$ and the relation given by the hypothesis of theorem leads to $p_\alpha(a_0 - a) \leqslant x^*(a_0 - a) + p_\alpha(a_0 - a)/n$ with $x^* \in K^0$ and $n \to \infty$, which implies that $p_\alpha(a_0 - a) \leqslant 0$, which is a contradiction and the proof is complete. □

REMARK 4.2. The above theorem represents an immediate extension of Precupanu's result given in Proposition 1.2 of [28]. In general, the converse of this theorem is not valid at least in (partially) ordered separated locally convex spaces as we can see from the example considered in Remark 4.1. Indeed, if one assumes the contrary in the corresponding, mathematical background, then, taking $\eta = \frac{1}{4}$ it follows that for each $\lambda_0 \in [0,1]$ there exists $c_1, c_2 \leqslant 0$ such that $|\lambda_0 - \lambda| \leqslant (c_1 - c_2)(\lambda_0 - \lambda) + \frac{1}{4}, \forall \lambda \in [0,1]$. Taking $\lambda = \frac{1}{4}$ one obtains $|1 - 4\lambda| \leqslant (c_1 - c_2)(1 - 4\lambda) + 1, \forall \lambda \in [0,1]$ which for $\lambda = 0$ implies that $c_2 \leqslant c_1$ and for $\lambda = \frac{1}{2}$ leads to $c_l \leqslant c_2$, that is, $|1 - 4\lambda| \leqslant 1, \forall \lambda \in [0,1]$, a contradiction.

The beginning and the consideration of Section 4 in [15] suggested us to consider for each function $\varphi : P \to K^* \backslash \{0\}$ the full nuclear cone.

$$K_\varphi = \{x \in X : p(x) \leqslant \varphi(p)(x), \forall p \in P\}$$

and to give the next generalization of Theorem 7 [15] in a more general context, which represents also a new important link between strong optimization and the approximate vector optimization together with its usual particular variant, respectively.

THEOREM 4.3. *If there exists* $\varphi_0 : P \to K^*\backslash\{0\}$ *with* $K \subseteq K_{\varphi 0}$ *then*

$$\textit{eff}(A, K, K_1) = \bigcup_{\substack{a \in A \\ \varphi \in P \to K^*\backslash\{0\}}} S(A \cap (a - K - K_1), K_\varphi)$$

for any non-empty subset K_1 *of* K.

Proof. If $a_0 \in \text{eff}(A, K, K_1)$ is an arbitrary element, then, in accordance with the point (i) of the Definition 2.1 and the hypothesis of the above theorem, we have $A \cap (a_0 - K - K_1) - a_0 \subseteq K + K_1 \subseteq K \subseteq K_{\varphi 0}$ for $\varphi_0 : P \to K^*\backslash\{0\}$ given by assumption. Therefore, $a_0 \in S(A \cap (a_0 - K - K_1), K_{\varphi_0})$. Hence,

$$\text{eff}(A, K, K_1) \subseteq \bigcup_{\substack{a \in A \\ \varphi \in P \to K^*\backslash\{0\}}} S(A \cap (a_0 - K - K_1), K_\varphi)$$

Conversely, let now $a_l \in S(A \cap (a_0 - K - K_l), K_\varphi)$ for at least one elements $a_0 \in A$ and $\varphi : P \to K^*\backslash\{0\}$. Then, $a_1 \in A \cap (a_0 - K - K_1)$ and $A \cap (a_0 - K - K_1) - a_1 \subseteq K_\varphi$ that is, $p(a - a_l) \leqslant \varphi(p)(a - a_1), \forall a \in A \cap (a_0 - K - K_1), p \in P$ which implies immediately that $p(a_1 - a) \leqslant \varphi(p)(a_1 - a) + \eta, \forall a \in A \cap (a_0 - K - K_1), p \in P, \eta \in (0, 1)$ and, by virtue of Theorem 4.2 one obtains $a_1 \in \text{eff}(A \cap (a_0 - K - K_1), K, K_1)$. But $\text{eff}(A \cap (a_0 - K - K_1), K, K_l) \subseteq \text{eff}(A, K, K_1)$.

Indeed, for any $t \in \text{eff}(A \cap (a_0 - K - K_1), K, K_1)$ and $h \in A \cap (t - K - K_1)$ we have $h \in A \cap (a_0 - K - K_1) \cap (t - K - K_1) \subseteq t + K + K_1$ that is, $A \cap (t - K - K_l) \subseteq t + K + K_1$ and by point (i) of Definition 2.1 one obtains $t \in \text{eff}(A, K, K_1)$. This completes the proof. □

REMARK 4.3. The hypothesis $K \subseteq K_{\varphi_0}$ imposed upon the convex cone K is automatically satisfied whenever K is a supernormal (nuclear) cone and it was used only to prove the inclusion $\text{eff}(A, K, K_1) \subseteq \bigcup_{\substack{a \in A \\ \varphi \in P \to K^*\backslash\{0\}}} S(A \cap (a - K - K_1), K_\varphi)$. When K is any pointed convex cone, A is a non-empty subset of X and $a_0 \in \text{eff}(A, K)$, then, by virtue of (i) in Remark 2.3, it follows that $A \cap (a_0 - K) = \{a_0\}$ that is, $A \cap (a_0 - K) - a_0 = \{0\} \subset K_\varphi$. Hence, $a_0 \in S(A \cap (a_0 - K), K_\varphi)$ for every mapping $\varphi : P : K^*\backslash\{0\}$ and the next corollary is valid.

COROLLARY 2.3.1. *For every non-empty subset* A *of any Hausdorff locally convex space ordered by an arbitrary, pointed convex cone* K *with its dual cone* K^* *we have*

$$\text{eff}(A, K) = \bigcup_{\substack{a \in A \\ \varphi: P \to K^* \setminus \{0\}}} S(A \cap (a - K), K_\varphi)$$

Comments

Clearly, the announced theorem represents a significant result concerning the possibilities of scalarization for the study of Pareto efficiency in Hausdorff locally convex spaces, as we can see also in the final comments of [15] for the particular cases of Hausdorff locally convex spaces ordered by closed, pointed and normal cones.

We note also that Proposition 3.1 and Theorem 3.2 proved in [21] support the fact that the open problem defined [15] seems to be interesting problem.

Because in a normed vector space a pointed closed convex cone is nuclear, if and only if the cone is well based, our results are more interesting in a general locally convex space. Considering Theorem 3.2 proved in [21], we have that, in our Theorem 4.3, replacing the norm by an equivalent norm, we can replace the cone K_φ, by the initial cone K. We note also that Theorems 4.2 and 4.3 are valid if we replace the spectrum P of the space X by a base of this spectrum. In this case in the formula given in Theorem 4.3, the union is reduced to a set.

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